XIX. *On the Algebraic Expression of the number of Partitions of which a given number is susceptible.* By Sir J. F. W. Herschel, *Bart., K.H., F.R.S.*

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(1.) Before entering on the investigation which forms the object of this communication, it will be necessary to recall to recollection some general properties of the differences of the powers of the natural numbers, or of the numbers comprised in the general expression $\Delta^m\, 0^n$, which I have elsewhere demonstrated, as well as to establish certain preliminary theorems by the aid of those properties, which will be useful in the progress of the inquiry. I shall employ throughout the separation of the symbols of operation from those of quantity, as respects Δ and 0, in the manner followed in my paper "On the Development of Exponential Functions," published in the Philosophical Transactions, vol. cvi. p. 25 (1816), and further extended in its application in my "Collection of Examples of the Applications of the Calculus of Finite Differences," appended to the translation of Lacroix's Differential and Integral Calculus in 1820*, to which paper and collection the reader is referred for the demonstration of the fundamental properties in question.

(2.) Denoting by $F(x)$ any series of powers of x, such as

$$F(x)=Ax^a+Bx^b+Cx^c+\&c.,$$

and by $f(x)$ any other as

$$f(x)=Px^p+Qx^q+Rx^r+\&c.$$

the series

$$AP.\Delta^a0^p+AQ.\Delta^a0^q+BP.\Delta^b0^p+\&c.,$$

continued till the terms vanish, by reason of the peculiar properties of the numbers Δ^a0^p &c., will be abbreviatively represented by

$$F(\Delta)f(0);$$

and the following properties of the differences in question will be either found demonstrated in the works above cited, or may very easily be derived from the formulæ therein given:—

$$(1+\Delta)^x 0^y = x^y \quad \ldots \quad (1.)$$

$$(1+\Delta)^x F(\Delta)0^y = F(\Delta)(x+0)^y \quad \ldots \quad (2.)$$

$$(1+\Delta)^x F(\Delta)f(0) = F(\Delta)f(x+0) \quad \ldots \quad (3.)$$

$$(1+\Delta)^x f(0) = f(x) \quad \ldots \quad (4.)$$

$$\{\log(1+\Delta)\}^x F(\Delta)0^y = y.\overline{y-1}\ldots\ldots\overline{y-x+1}.F(\Delta)0^{y-x} \quad \ldots \quad (5.)$$

$$\{\log(1+\Delta)\}^x F(\Delta)f(0) = F(\Delta)\left(\frac{d}{d0}\right)^x f(0) \quad \ldots \quad (6.)$$

* A separate edition of this collection (now out of print) is in preparation.

(3.) Furthermore, if we observe that

$$1-(1+\Delta)^{p+q}=\{1-(1+\Delta)^{p}\}+(1+\Delta)^{p}\{1-(1+\Delta)^{q}\},$$

we shall have, by applying each of these operative symbols to $F(\Delta)f(0)$,

$$\{1-(1+\Delta)^{p+q}\}F(\Delta)f(0)=\{1-(1+\Delta)^{p}\}F(\Delta)f(0)+\{1-(1+\Delta)^{q}\}F(\Delta)f(p+0)\,; \quad (7.)$$

and therefore

$$\{1-(1+\Delta)^{p}\}F(\Delta)f(q+0)-\{1-(1+\Delta)^{q}\}F(\Delta)f(p+0)=\{(1+\Delta)^{p}-(1+\Delta)^{q}\}F(\Delta)f(0); \quad (8.)$$

(4.) In the particular case where $F(\Delta)=\frac{1}{\Delta}$, these become

$$\frac{1-(1+\Delta)^{p+q}}{\Delta}f(0)=\frac{1-(1+\Delta)^{p}}{\Delta}f(0)+\frac{1-(1+\Delta)^{q}}{\Delta}f(p+0)\,; \quad . \quad . \quad . \quad . \quad (9.)$$

and

$$\frac{1-(1+\Delta)^{p}}{\Delta}f(q+0)-\frac{1-(1+\Delta)^{q}}{\Delta}f(p+0)=\frac{(1-\Delta)^{p}-(1+\Delta)^{q}}{\Delta}f(0). \quad . \quad . \quad (10.)$$

(5.) Designating by $S(x^n)$ the sum of the nth powers of the natural numbers from 1 to x inclusive, or putting

$$S(x^n)=1^n+2^n+\dots\dots x^n,$$

it is demonstrated (Examples, § 6. Ex. 23.) that

$$S(x^n)=(-1)^n.\frac{1-(1+\Delta)^{-x}}{\Delta}0^n=(-1)^n.\left\{\frac{x}{1}.0^n-\frac{x(x+1)}{1.2}\Delta0^n+\frac{x(x+1)(x+2)}{1.2.3}\Delta^2 0^n-\&c.\right\}; \quad . \quad . \quad (11.)$$

and again, in § 8, Exp. 11 of the same work, that

$$S(x^n)=(1+\Delta).\frac{(1+\Delta)^x-1}{\Delta}0^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

(6.) Furthermore, it will be necessary to recall in what follows, the notation and conventions of 'circulating functions,' as explained in my paper on that subject, published in the Philosophical Transactions for 1818, vol. cviii. p. 144. Denoting by s_x the sum of the xth powers of the sth roots of unity divided by s, or the function

$$\frac{1}{s}(\alpha^x+\beta^x+\gamma^x+\&c.),$$

where α, β, γ, &c. are those roots, any function of the form

$$A_x.s_x+B_x.s_{x-1}+\dots.N_x.s_{x-s+1}$$

will circulate in its successive values as x increases by units from 0: being expressed by A_x when x is a multiple of s; by B_x, when $x-1$ is such a multiple, and so on. If A_x, B_x, &c. be simply constant, the function may be termed a *periodic* one, since it assumes in periodic and constantly recurring succession the values A, B, C....N, A, B, &c. *ad infinitum*. If s be a specified number, as 2, 3, &c., we shall not the less use the notations 2_x, 3_x, &c. to express the respective quantities $\frac{1}{2}(\alpha^x+\beta^x)$, $\frac{1}{3}(\alpha^x+\beta^x+\gamma^x)$, &c., where α, β, &c. are the corresponding roots of unity. And we shall accordingly have the following general relations, in which P_x and Q_x denote any circulating functions, such that

$$P_x = a_x . s_x + b_x . s_{x-1} + c_x . s_{x-2} + \&c...,$$

$$Q_x = A_x . s_x + B_x . s_{x-1} + C_x s_{x-2} + \&c.$$

$$f(P_x, Q_x) = f(a_x, A_x) . s_x + f(b_x, B_x) . s_{x-1} + \&c., \quad \ldots \quad (13.)$$

of which, particular cases are

$$f(P_x) = f(a_x) . s_x + f(b_x) . s_{x-1} + f(c_x) . s_{x-2} + \&c. \quad \ldots \quad (14.)$$

$$P_x^n = a_x^n . s_x + b_x^n s_{x-1} + c_x^n . s_{x-2} + \&c. \quad \ldots \quad (15.)$$

$$Q_x^{P_x} = Q_x^{a_x} . s_x + Q_x^{b_x} s_{x-1} + Q_x^{c_x} . s_{x-2} + \&c. \quad \ldots \quad (16.)$$

$$P_x Q_x = a_x . A_x . s_x + b_x . B_x . s_{x-1} + \&c. \quad \ldots \quad (17.)$$

(7.) As special relations to which we shall refer, we have

$$s_x + s_{x-1} + \ldots\ldots s_{x-s+1} = 1, \quad \ldots \quad (18.)$$

and since also (y being any other index)

$$t_y + t_{y-1} + \ldots . t_{y-t+1} = 1.$$

Therefore, multiplying and denoting by S the sum of all terms so originating,

$$S\{s_{x-i} . t_{y-j}\} = 1, \quad \ldots \quad (19.)$$

i having all values from $i=0$ to $i=s-1$, and j all values from $j=0$ to $j=t-1$. And the same holds good for any number of indices x, y, z, &c.

(8.) If n and s be prime to one another, we shall also have

$$n_x + n_{x-s} + n_{x-2s} + \ldots . n_{x-(n-1)s} = 1. \quad \ldots \quad (20.)$$

For if the series of numbers 0, s, $2s$, ... $(n-1)s$ be divided by n, they will leave s remainders, all different, and all less than n, so that among them will be found, though not in the same order, all the numbers 0, 1, 2, ... $(n-1)$, whence, since

$$n_{x-i} = n_{x-nm-i},$$

the truth of the equation (20.) is apparent.

(9.) If n be a multiple of s, or $n=ts$, then

$$n_x + n_{x-s} + n_{x-2s} + \ldots n_{x-\overline{t-1}.s} = s_x. \quad \ldots \quad (21.)$$

But if n and s have a common measure v, so that $n=tv$, $s=qv$, then

$$n_x + n_{x-s} + \ldots . n_{x-ts+s} = v_x. \quad \ldots \quad (22.)$$

Thus, for example,

$$6_x + 6_{x-2} + 6_{x-4} = 2_x ; \quad 6_x + 6_{x-3} = 3_x$$

$$15_x + 15_{x-6} + \ldots . 15_{x-24} = 3_x.$$

(10.) To find the product or other functional combination of circulating or periodic functions having different periods of circulation, they must be reduced to a common period. Thus, if m represent the product of s and t divided by their greatest common measure, and the functions to be combined be

$$P_x = a_x . s_x + b_x . s_{x-1} + \&c.$$

$$Q_x = A_x . t_x + B_x . t_{x-1} + \&c.,$$

we have by equation (22.),

$$P_x = a_x . m_x + b_x . m_{x-1} + \dots + a_x . m_{x-s} + b_x . m_{x-s-1} + \dots \&c.$$

$$Q_x = A_x . m_x + B_x . m_{x-1} + \dots + A_x . m_{x-t} + B_x . m_{x-t-1} + \dots \&c.,$$

and consequently

$$f\{P_x, Q_x\} = f\{a_x, A_x\} . m_x + f\{b_x, B_x\} m_{x-1} + \&c. \dots$$

For example,

$$(a.2_x + b.2_{x-1})(A.3_x + B.3_{x-1} + C.3_{x-2}) = aA.6_x + bB.6_{x-1} + aC.6_{x-2} + bA.6_{x-3} + aB.6_{x-4} + bC.6_{x-5}.$$

(11.) The readiest way in practice and the surest way to avoid mistakes, which in complex cases are very likely to occur, is to proceed by a much easier process, as in the following example. Suppose we would express the product of the three periodic functions

$$P_x = 2_x + 2.2_{x-1}, \quad Q_x = 3_x + 2.3_{x-1} + 3.3_{x-2}, \quad R_x = 4_x + 2.4_{x-1} + 3.4_{x-2} + 4.4_{x-3},$$

the product of 2, 3, 4 divided by the greatest common measure of 2 and 4 is 12, which will therefore be the period of the product. Write then the several coefficients in order as follows: for

P_x	1,	2;	1,	2;	1,	2;	1,	2;	1,	2;	1,	2;	&c.
Q_x	1,	2,	3;	1,	2,	3;	1,	2,	3;	1,	2,	3;	&c.
R_x	1,	2,	3,	4;	1,	2,	3,	4;	1,	2,	3,	4;	&c.
	1,	8,	9,	8,	2,	12,	3,	16,	3,	4,	6,	24,	

and we shall have for the product

$$12_x + 8.12_{x-1} + 9.12_{x-2} + 8.12_{x-3} + 2.12_{x-4} + \dots\dots 24.12_{x-11}.$$

If the signs of the coefficients, or any of them, differ, they must be of course annexed to each, and the proper sign affixed to each product.

(12.) If we denote by $\underset{s}{\overset{x}{=}}$ the integer part of the quotient of any number x divided by another s, then, universally,

$$\underset{s}{\overset{x}{=}} = \frac{x}{s} . s_x + \frac{x-1}{s} s_{x-1} + \dots \frac{x-s+1}{s} s_{x-s+1}; \quad \dots \quad (23.)$$

$$= \frac{x}{s} - \frac{1}{s}\left\{0.s_x + 1.s_{x-1} + 2.s_{x-2} + \dots \overline{s-1}.s_{x-s+1}\right\}$$

and therefore the remainder, in the same division, is expressed by

$$x - s.\underset{s}{\overset{x}{=}} = 0.s_x + 1.s_{x-1} + 2.s_{x-2} + \dots \overline{s-1}.s_{x-s+1}. \quad \dots \quad (24.)$$

(13.) If, again, we would express the integer part of the quotient in the division of

$\frac{x}{s}$ by a second integer division t, we have, putting y for $\frac{x}{s}$,

$$\frac{y}{t}=\frac{y}{t}-\frac{1}{t}\left\{0.t_y+1.t_{y-1}+\ldots(t-1).t_{y-t+1}\right\}$$
$$=\frac{x}{st}-\frac{1}{st}\left\{0.s_x+1.s_{x-1}+\ldots.(s-1)s_{x-s+1}\right\}$$
$$-\frac{1}{t}\left\{0.t_y+1.t_{y-1}+\ldots\ldots(t-1).t_{y-t+1}\right\}; \qquad (25.)$$

and so on as far as we please.

(14.) The periodic function $0.t_y+1.t_{y-1}+$ &c. depends implicitly on x, because y is dependent on x. Its value however (as well as that of any other implicit periodical function) is very easily obtained by following out the process explained in (art. 11). Suppose, for example, we had the more general periodic function of y,

$$q_y=a.t_y+b.t_{y-1}+\ldots.h.t_{y-t+1}.$$

Then we may write down the successive values of x, y, q_y in order thus:

x	0, 1, 2, ... $(s-1)$;	s, $s+1$, ... $(2s-1)$;	$2s$, ... ;	ts, ...
y	0, 0, 0, ... 0 ;	1, 1, ... 1 ;	2 , ... ;	t , ...
q_y	a, a, a, ... a ;	b, b, ... b ;	c , ... ;	a , ...

Thus we see that q_y is a periodical function of x, having for its period st instead of either s or t separately, the first s coefficients being all alike and each $=a$, the next s all alike and each $=b$, and so on; or

$$q_y=a\{(st)_x+\ldots(st)_{x-s+1}\}+b\{(st)_{x-s}+\ldots\ldots(st)_{x-2s+1}\}+\ldots..+h\{(st)_{x-ts+s}+\ldots.(st)_{x-ts+1}\}. \qquad (26.)$$

(15.) Hence we are enabled to express the value of $\frac{y}{t}$ explicitly as a periodic function of x; for by equation (25.), if we put $st=n$,

$$\frac{y}{t}=\frac{x}{n}-\frac{1}{n}\left\{0.s_x+1.s_{x-1}+\ldots.\overline{s-1}.s_{x-s+1}\right\}$$
$$-\frac{1}{n}\left\{0.t_y+s.t_{y-1}+\ldots.s.\overline{t-1}.t_{y-t+1}\right\}.$$

But by equation (21.) we have

$$s_x\ \ =n_x+n_{x-s}+n_{x-2s}+\ldots n_{x-ns+s}$$
$$s_{x-1}=n_{x-1}+n_{x-s-1}+\ldots\ldots n_{x-ns+s-1}, \text{ \&c.};$$

and by the equation (26.) of the foregoing article,

$$t_y\ \ =n_x+n_{x-1}+\ldots.n_{x-s+1}$$
$$t_{y-1}=n_{x-s}+n_{x-s-1}+\ldots n_{x-2s+1}$$
$$t_{y-2}=n_{x-2s}+n_{x-2s-1}+\ldots.n_{x-3s+1}, \text{ \&c.};$$

and consequently by substitution,

$$\frac{y}{t}=\frac{x}{n}-\frac{1}{n}\left\{0.n_x+1.n_{x-1}+2.n_{x-2}+\ldots..\overline{n-1}.n_{x-n+1}\right\}. \qquad (27.)$$

(16.) These relations premised, let it be required to express the sum of x terms of the series

$$\varphi(a+b)+\varphi(a+2b)+\ldots\ldots\varphi(a+xb)=S_x.$$

Developing the several terms, we find

$$\begin{aligned} S_x=&\varphi(a)\{1+1+\ldots\ldots x \text{ terms}\}\\ &+\frac{b}{1}.\varphi'(a)\{1+2+3+\ldots\ldots x\}\\ &+\frac{b^2}{1.2}\varphi''(a)\{1^2+2^2+3^2+\ldots.x^2\}+\&c.\end{aligned}$$

Substituting then for the series of powers of 1, 2, 3, &c. their values as given by equation (11.), and separating the symbols of operation from that of quantity, we get

$$S_x=\frac{1-(1+\Delta)^{-x}}{\Delta}\left\{\varphi(a).0^0-\varphi'(a).\frac{b}{1}.0^1+\varphi''(a).\frac{b^2}{1.2}.0^2-\&c.\right\}=\frac{1-(1+\Delta)^{-x}}{\Delta}\varphi(a-b.0). \quad (28.)$$

(17.) If we use in like manner equation (12.), it gives

$$S_x=(1+\Delta)\frac{(1+\Delta)^x-1}{\Delta}\varphi(a+b.0)=\frac{(1+\Delta)^x-1}{\Delta}\varphi(\overline{a+b}+b.0), \quad . \quad . \quad . \quad (29.)$$

by employing the transformation of equation (3.), in which $x=1$, $f(0)=\varphi(a+b.0)$. Hence also, if

$$S_x=\varphi(a)+\varphi(a+b)+\ldots.\varphi(a+\overline{x-1}.b),$$

we find in like manner

$$S_x=\frac{(1+\Delta)^x-1}{\Delta}\varphi(a+b.0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30.)$$

(18.) Let it next be required to find the sum of the series

$$S_y=\varphi(a+b)+\varphi(a+2b)+\ldots.\varphi(a+yb)$$

to y terms, where $y=\frac{x}{s}$ the integer part of the quotient of an independent index number x, divided by any given number s. By equation (28.) we have

$$S_y=\frac{1-(1+\Delta)^{-y}}{\Delta}\varphi(a-b.0).$$

Now since

$$\begin{aligned} y=&\frac{x}{s}s_{x-1}+\frac{x-1}{s}s_{x-1}+\ldots\ldots\frac{x-s+1}{s}s_{x-s+1}\\ =&\frac{x}{s}-\frac{1}{s}\left\{0.s_x+1.s_{x-1}+\ldots.(s-1).s_{x-s+1}\right\}.\end{aligned}$$

If we put

$$p=-\frac{x}{s}; \quad q=\frac{1}{s}\left\{0.s_x+1.s_{x-1}+\&c.\right\},$$

we get by equation (9.),

$$S_y=\frac{1-(1+\Delta)^p}{\Delta}\varphi(a-b.0)+\frac{1-(1+\Delta)^q}{\Delta}\varphi(\overline{a-pb}-b.0).$$

Now the first portion of this, since $p=-\frac{x}{s}$, is explicit in terms of x, but the other requires further development, for which we must have recourse to equation (16.), putting $Q_x=(1+\Delta)$ and $P_x=q=\frac{1}{s}(0.s_x+1.s_{x-1}+\&c.)$, where we find

$$(1+\Delta)^q=(1+\Delta)^0.s_x+(1+\Delta)^{\frac{1}{s}}.s_{x-1}+(1+\Delta)^{\frac{2}{s}}.s_{x-2}+\&c.$$

But we also have by equation (18.),

$$1=\quad s_x+\quad s_{x-1}+\quad s_{x-2}+\&c.$$

Therefore, subtracting and dividing by Δ, and applying each term of operation to the term $\varphi(\overline{a-pb}-b.0)$ of quantity,

$$\frac{1-(1+\Delta)^q}{\Delta}\varphi(\overline{a-pb}-b.0)=\left\{0.s_x+\frac{1-(1+\Delta)^{\frac{1}{s}}}{\Delta}s_{x-1}+\frac{1-(1+\Delta)^{\frac{2}{s}}}{\Delta}s_{x-2}+\&c.\right\}\varphi(\overline{a-pb}+b.0)$$

$$=0.s_x+\frac{1-(1+\Delta)^{\frac{1}{s}}}{\Delta}\varphi(\overline{a-pb}-b.0).s_{x-1}+\frac{1-(1+\Delta)^{\frac{2}{s}}}{\Delta}\varphi(\overline{a-pb}-b.0).s_{x-2}+\&c. \quad . \quad (31.)$$

(19.) The expression for S_y in the last article is general and entirely independent of any particular values assigned to x, a, b, s, the only relation established being that expressed by the equation $y=\frac{x}{s}$. Suppose therefore that in a certain proposed case we should have

$$a=x+s-1\,;\quad b=-s\,;$$

and therefore

$$pb=x\,;\quad a-pb=s-1,$$

and the expression for the sum of the series becomes

$$S_y=\frac{1-(1+\Delta)^{-\frac{x}{s}}}{\Delta}\varphi(x+s-1+s.0)$$

$$+\frac{1-(1+\Delta)^{\frac{1}{s}}}{\Delta}\varphi(\overline{s-1}+s.0).s_{x-1}$$

$$+\frac{1-(1+\Delta)^{\frac{2}{s}}}{\Delta}\varphi(\overline{s-1}+s.0).s_{x-2}+\&c.\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32.)$$

in which expression the first member or non-periodical part is an explicit function of x and s, and the periodical part has all its coefficients independent on x and functions of s alone.

(20.) The periodical part of S_y is however susceptible of another form, better adapted for numerical calculation, into which it may be thrown by making $p=q=\frac{1}{s}$ in equation (9.), when it becomes

$$\frac{1-(1+\Delta)^{\frac{2}{s}}}{\Delta}f(0)=\frac{1-(1+\Delta)^{\frac{1}{s}}}{\Delta}f(0)+\frac{1-(1+\Delta)^{\frac{1}{s}}}{\Delta}f\left(\frac{1}{s}+0\right),$$

in which, for $f(0)$ writing $\varphi(\overline{s-1}+s.0)$, it becomes

$$\frac{1-(1+\Delta)^{\frac{2}{s}}}{\Delta}\varphi(\overline{s-1}+s.0)=\frac{1-(1+\Delta)^{\frac{1}{s}}}{\Delta}\left\{\varphi(s-1+s.0)+\varphi(s+s.0)\right\}.$$

Similarly,

$$\frac{1-(1+\Delta)^{\frac{3}{s}}}{\Delta}\varphi(s-1+s.0)=\frac{1-(1+\Delta)^{\frac{1}{s}}}{\Delta}\left\{\varphi(\overline{s-1}+s.0)+\varphi(s+s.0)+\varphi(\overline{s+1}+s.0)\right\},$$

and so on; the general form assumed by our equation (9.), on applying this process, being

$$\frac{1-(1+\Delta)^{np}}{\Delta}f(0)=\frac{1-(1+\Delta)^{p}}{\Delta}\left\{f(0)+f(p+0)+f(2p+0)\ldots.+f(\overline{n-1}.p+0)\right\} \quad . \quad . \quad (33.)$$

Supposing, then, for brevity, we denote the combination $\frac{1-(1+\Delta)^{\frac{1}{s}}}{\Delta}$ by ∇, and that we put

$$\psi_1(s)=\varphi(s-1)\ ;\ \psi_2(s)=\varphi(s-1)+\varphi(s)\ ;\ \psi_3(s)=\varphi(s-1)+\varphi(s)+\varphi(s+1)\ ;\ \&c.,$$

then we shall have, finally,

$$\begin{aligned}S_y=\frac{1-(1+\Delta)^{-\frac{x}{s}}}{\Delta}\varphi\left\{\overline{x+s-1}+s.0\right\}&+0.s_x+s_{x-1}.\nabla\psi_1(s+0.s)\\&+s_{x-2}.\nabla\psi_2(s+0.s)\\&+\&c.; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34.)\\=X+Y,\end{aligned}$$

where X represents the non-periodical part, a function of x, and Y the periodical, whose coefficients are constant.

(21.) For the actual evaluation of these functions, all we have to do is to develope the operative characteristic in powers of Δ, and the attached functions in powers of 0, and to apply them term by term to each other. As regards the function X, we find, by so doing,

$$\begin{aligned}X=\frac{1}{s}\Big\{\frac{x}{1}.\varphi(x+s-1)-\frac{x(x+s)}{1.2}.\Delta 0.\frac{\varphi'(x+s-1)}{1}&\\+\left\{\frac{x(x+s)(x+2s)}{1.2.3}.\Delta^2 0^2-\frac{x(x+s)}{1.2}.s.\Delta 0^2\right\}.\frac{\varphi''(x+s-1)}{1.2}&\\-\left\{\frac{x(x+s)\ldots(x+3s)}{1\ldots 4}\Delta^3 0^3-\frac{x(x+s)(x+2s)}{1.2.3}.s\Delta^2 0^3+\frac{x(x+s)}{1.2}.s^2.\Delta 0^3\right\}.\frac{\varphi'''(x+s-1)}{1.2.3}&\\+\&c., \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35.)&\end{aligned}$$

in which it will be recollected that

$$\Delta 0=1\ ;\ \Delta 0^2=1,\ \Delta^2 0^2=2\ ;\ \Delta 0^3=1,\ \Delta^2 0^2=6\ ;\ \Delta^3 0^3=6\ ;\ \&c.\ \&c.$$

(22.) In like manner, denoting by $\psi(s)$ in general, any of the functions $\psi_1(s)$, $\psi_2(s)$,

&c., we have

$$\nabla\psi(s+0.s)=-\frac{1}{s}\Big\{\psi(s)-\frac{s-1}{1.2}\Delta 0.\frac{\psi'(s)}{1}+\Big(\frac{\overline{s-1}.\overline{2s-1}}{1.2.3}\Delta^2 0^2-\frac{s-1}{1.2}.s.\Delta 0^2\Big).\frac{\psi''(s)}{1.2}$$
$$-\Big(\frac{\overline{s-1}.\overline{2s-1}.\overline{3s-1}}{1.2.3.4}.\Delta^3 0^3-\frac{\overline{s-1}.\overline{2s-1}}{1.2.3}.s\Delta^2 0^3+\frac{s-1}{1.2}.s^2.\Delta 0^3\Big).\frac{\psi'''(s)}{1.2.3}$$
$$+\&c.\Big\}. \qquad (36.)$$

Hence if we put

$$\Psi(s)=0.s_x+\psi_1(s).s_{x-1}+\psi_2(s).s_{x-2}+\ldots\ldots\psi_{s-1}(s)s_{x-s+1},$$

and denote by $\Psi'(s)$, $\Psi''(s)$, &c. the differential coefficients of $\Psi(s)$, *regarding the discontinuous functions* s_x, s_{x-1}, &c. *as incapable of differentiation*, we shall have, finally,

$$Y=-\frac{1}{s}\Big\{\Psi(s)-\frac{s-1}{1.2}\Delta 0.\frac{\Psi'(s)}{1}+\Big\{\frac{\overline{s-1}.\overline{2s-1}}{1.2.3}\Delta^2 0^2-\frac{s-1}{1.2}.s.\Delta 0^2\Big\}.\frac{\Psi''(s)}{1.2}$$
$$-\Big\{\frac{\overline{s-1}\ldots\overline{3s-1}}{1\ldots.4}\Delta^3 0^3-\frac{\overline{s-1}.\overline{2s-1}}{1.2.3}.s.\Delta^2 0^3+\frac{s-1}{1.2}s^2.\Delta 0^3\Big\}\frac{\Psi'''(s)}{1.2.3}$$
$$+\&c.\Big\} \qquad (37.)$$

(23.) As regards s, in every part of the following investigation it will be regarded as a mere given integer number, so that the coefficients in s will come to be calculated in absolute numbers and need give no further concern. It is otherwise with those in x, which mix themselves up with the x contained in $\phi(x+s-1)$ and its differential coefficients in a way requiring special examination, as functions of an independent variable. Let us therefore consider the term multiplied by $\left(\frac{d}{dx}\right)^n\phi(x+s-1)$ in the development of $\phi(x+\overline{s-1}-s.0)$. This term will be expressed in Arbogast's notation by

$$\frac{s^n.0^n}{1.2\ldots..n}D^n\phi(x+s-1),$$

and the corresponding term in X will be

$$\frac{s^n}{1.2\ldots n}.D^n\phi(x+s-1).\frac{1-(1+\Delta)^{-\frac{x}{s}}}{\Delta}0^n.$$

The development of this in factorials x, $x+s$, $x+2s$, &c., and functions ϕ, ϕ', ϕ'', &c., is accomplished in equation (35.), but if we would effect it in powers of x we must proceed as follows :—

Suppose $e=2{\cdot}7182818$, &c. Then we have

$$\frac{1-(1+\Delta)^{-\frac{x}{s}}}{\Delta}0^n=\frac{1-e^{-\frac{x}{s}.\log(1+\Delta)}}{\Delta}0^n$$
$$=\frac{x}{1.s}.\frac{\log(1+\Delta)}{\Delta}0^n-\frac{x^2}{1.2.s^2}.\frac{\{\log(1+\Delta)\}^2}{\Delta}0^n+\frac{x^3}{1.2.3}.\frac{\{\log(1+\Delta)\}^3}{\Delta}0^n-\&c.$$

But by equation (5.),

$$\frac{\{\log(1+\Delta)\}^{i+1}}{\Delta}0^n=\{\log(1+\Delta)\}^i.\frac{\log(1+\Delta)}{\Delta}0^n$$

$$=n(n-1)\ldots.(n-i+1).\frac{\log(1+\Delta)}{\Delta}0^{n-i},$$

and therefore the foregoing expression becomes

$$\frac{x}{s}.\frac{\log(1+\Delta)}{\Delta}0^n-\frac{n}{2}\left(\frac{x}{s}\right)^2.\frac{\log(1+\Delta)}{\Delta}0^{n-1}+\frac{n(n-1)}{2.3}.\&c.;$$

or, inverting the order of the terms,

$$(-1)^n.\left\{\frac{1}{n+1}\left(\frac{x}{s}\right)^{n+1}.\frac{\log(1+\Delta)}{\Delta}0^0-\left(\frac{x}{s}\right)^n.\frac{\log(1+\Delta)}{\Delta}0^1+\right.$$

$$\left.+\frac{n}{2}\left(\frac{x}{s}\right)^{n-1}.\frac{\log(1+\Delta)}{\Delta}0^2-\frac{n(n-1)}{2.3}\left(\frac{x}{s}\right)^{n-2}.\frac{\log(1+\Delta)}{\Delta}0^3+\&c.\right\}.$$

Now if B_1, B_3, B_5, &c. be the numbers of BERNOULLI in their order (the even values B_2, B_4, &c. being severally $=0$), we have

$$\frac{\log(1+\Delta)}{\Delta}0^0=1;\ \frac{\log(1+\Delta)}{\Delta}0^1=-\frac{1}{2};\ \frac{\log(1+\Delta)}{\Delta}0^2=B_1=\frac{1}{6},$$

and so on. Consequently

$$\frac{1-(1+\Delta)^{-\frac{x}{s}}}{\Delta}0^n=(-1)^n.\left\{\frac{1}{n+1}\left(\frac{x}{s}\right)^{n+1}+\frac{1}{2}\left(\frac{x}{s}\right)^n+\frac{n}{2}B_1\left(\frac{x}{s}\right)^{n-1}+\right.$$

$$\left.+\frac{n(n-1)}{2.3}.B_2\left(\frac{x}{s}\right)^{n-2}+\frac{n(n-1)(n-2)}{2.3.4}B_3.\left(\frac{x}{s}\right)^{n-3}+\&c.\right\}, \quad \ldots \ldots \ldots \quad (38.)$$

the series on the right-hand side being continued to $n+1$ terms. This is in fact no other than EULER's expression for the sum of the series $1^n+2^n+3^n+$ &c. to a given number of terms represented by $\frac{x}{s}$, only that in the case here under consideration $\frac{x}{s}$ may be any fraction, while EULER's demonstration of the series in question is essentially confined to $\frac{x}{s}=$ a positive integer.

(24.) If we make $x=-1$, the foregoing expression becomes

$$\frac{1-(1+\Delta)^{\frac{1}{s}}}{s}0^n=\nabla 0^n=-\frac{1}{n+1}\left(\frac{1}{s}\right)^{n+1}+\frac{1}{2}\left(\frac{1}{s}\right)^n-\frac{n}{2}B_1.\left(\frac{1}{s}\right)^{n-1}-\frac{n(n-1)}{2.3}.B_2\left(\frac{1}{s}\right)^{n-2}+\&c.$$

$$=-\frac{1}{s^{n+1}}\left\{\frac{1}{n+1}-\frac{1}{2}.s+\frac{n}{2}B_1.s^2-\frac{n(n-1)}{2.3}B_2.s^3+\&c.\right\} \quad \ldots \ldots \ldots \quad (39.)$$

continued to $n+1$ terms, inclusive of the vanishing ones having B_2, B_4, &c. for coefficients.

(25.) Thus the expression for X becomes

$$X=\frac{x}{s}.\phi(x+s-1)-\frac{x^2+sx}{2.s}\phi'(x+s-1) \quad \ldots \ldots \quad (40.)$$

$$+\frac{2x^3+3sx^2+s^2x}{12.s}\phi''(x+s-1)$$

$$-\frac{x^4+2sx^3+s^2x^2}{24.s}\phi'''(x+s-1)$$

$$+\frac{6x^5+15x^4s+10s^2x^3+s^4x}{720.s}\phi^{\text{IV}}(x+s-1)-\&c.$$

$$Y=-\frac{1}{s}\Big\{\Psi(s)-\frac{s-1}{2}.\Psi'(s)+\frac{s^2-3s+2}{12}.\Psi''(s)+\frac{s^2-2s+1}{24}\Psi'''(s)+$$

$$+\frac{s^4+10s^2-15s+6}{720}.\Psi^{\text{IV}}(s)+\&c.\Big\} \quad \ldots \ldots \quad (41.)$$

(26.) We shall now proceed to the more immediate object of this paper, viz. the expression of the number of ways in which a given number x is susceptible of partition, the number of parts being given.

Let s be the number of parts into which x is to be divided, and let ${}^s\Pi(x)$ represent the number of s-partitions of which it is susceptible. It is evident then that if $s=1$ there is but one possible, so that in all cases ${}^1\Pi(x)=1$.

If $s=2$, the partitions stand thus:

$$1, x-1\ ; \quad 2, x-2\ ; \quad 3, x-3\ ; \ \ldots.$$

whose number is $\frac{x}{2}$. Therefore we have

$${}^2\Pi(x)=\frac{x}{2}=\frac{x}{2}-\frac{1}{2}\cdot 2_{x-1}.$$

If $s=3$, the partitions so grouped as that none shall be twice repeated, will stand as follows:—

1, 1, $x-2$	2, 2, $x-4$	3, 3, $x-6$, &c.
1, 2, $x-3$	2, 3, $x-5$	&c.
1, 3, $x-4$	&c.	
&c.		

The first column will consist of all the possible bipartitions of $x-1$, each associated with 1, and their number is therefore ${}^2\Pi(x-1)$. The second will consist of the bipartitions of $x-2$, exclusive of $(1, x-3)$, each associated with 2. Their number therefore will be identical with the total number of bipartitions of $x-4$, because, so far as the number of cases is concerned, it matters not whether we consider $x-4$ as parted into $(1, x-5)$, $(2, x-6)$, &c., or $x-2$ as parted into $(2, x-4)$, $(3, x-5)$, &c., the reason of which will be obvious on trying any particular case. The number of terms therefore in the second column will be ${}^2\Pi(x-4)$. In like manner that in the third will be ${}^2\Pi(x-7)$, the bipartitions of $x-3$ beginning with $(3, x-6)$, being identical in number with those of $x-7$, beginning with $(1, x-8)$, and so on. Thus we have

$$^{3}\Pi(x)={}^{2}\Pi(x-1)+{}^{2}\Pi(x-4)+{}^{2}\Pi(x-7)+\&c.$$

Next, with respect to the number of terms to which the right-hand member of this equation is to be continued: it will be that of the columns, which will continue without repetition so long as the number $x-2m$ in the first combination $(m, m, x-2m)$ of any one of them shall be not less than m, or so long as $x-3m$ shall not be negative. Hence we must have $m=\frac{x}{3}$, since the next greater value of m, viz. $m=\frac{x}{3}+1$, will give the tripartition $\left(\frac{x}{3}+1, \frac{x}{3}+1, x-2\frac{x}{3}-2\right)$. Now x cannot exceed $3.\frac{x}{3}$ by more than 2, so that $x-2\frac{x}{3}-2$ cannot exceed $\frac{x}{3}$, and must therefore be less than $\frac{x}{3}+1$. Hence we conclude that the number of tripartitions is derived from that of bipartitions by the equation

$$^{3}\Pi(x)={}^{2}\Pi(x-1)+{}^{2}\Pi(x-4)+\ldots{}^{2}\Pi(x-7)\ldots\text{ to }\frac{x}{3}\text{ terms.}$$

(27.) Applying a similar reasoning to the higher cases, we shall find as follows:—

$$\overset{s}{\Pi}(x)=\overset{s-1}{\Pi}(x-1)+\overset{s-1}{\Pi}(x-s-1)+\overset{s-1}{\Pi}(x-2s-1)\ldots\frac{x}{s}\text{ terms;} \quad \ldots \quad (42.)$$

a relation which, with many others of greater generality, has also been arrived at by Mr. WARBURTON.

(28.) Suppose now we set out from the equation $^{1}\Pi(x)=1$, and proceed to derive from this value those of $^{2}\Pi(x)$, $^{3}\Pi(x)$, &c. in succession. It will be apparent from the course of the foregoing investigations, and from the nature of circulating functions, that the general expression for $\overset{s}{\Pi}(x)$ must consist of two portions, the one non-periodical, a function of x, and which may be represented by $\varphi(x)$, the other periodical or circulating, which we may denote by Q_x, so that we shall have in general to consider the following form of $\overset{s-1}{\Pi}(x)$,

$$\overset{s-1}{\Pi}(x)=\varphi(x)+Q_x,$$

from which to derive the value of $\overset{s}{\Pi}(x)$.

When we substitute this in the general expression (equation 42.), we get

$$\overset{s}{\Pi}(x)=\varphi(x-1)+\varphi(x-s-1)+\ldots\ldots(y\text{ terms})$$
$$+Q_{x-1}+Q_{x-s-1}+\ldots\ldots(y\text{ terms}),$$

where $y=\frac{x}{s}$. Now with respect to the first portion of this, if $\varphi(x)$ in any one case be a rational integral function of x, it will be so in all subsequent cases, as is evident from the course of the preceding investigations. This part of $\overset{s}{\Pi}(x)$ then has been already dealt with, and its complete expression is X+Y of equations (34. 35.), or (38. 39.).

(29.) We have therefore now only to consider the remaining portions, which we shall call Z, viz.

$$Z=Q_{x-1}+Q_{x-s-1}+\dots y \text{ terms},$$

Q_x may represent any circulating function. Suppose it to be such that

$$Q_{x-1}=\chi_0(x).m_x+\chi_1(x).m_{x-1}+\dots\dots\chi_{m-1}(x).m_{x-m+1},$$

and let any term of this, as $\chi_i(x)m_{x-i}$ (which for brevity we will write simply $\chi(x).m_z$, putting $z=x-i$), be separately considered. Let R be the portion of Z which originates from this term. Then

$$R=\chi(x).m_z+\chi(x-s).m_{z-s}+\chi(x-2s).m_{z-2s}\dots\dots(y \text{ terms}).$$

Let $ts=n$ be the first multiple of s, which is also a multiple of m. Then after t terms the value of m_z, m_{z-s}, &c. will recur, and therefore R resolves itself into t separate series, as follows:

$$\begin{aligned}R=m_z&\left\{\chi(x)+\chi(x-n)+\chi(x-2n)+\dots\left(\overline{\overline{\tfrac{y-1}{t}}}+1\right)\text{terms}\right\}\\ +m_{z-s}&\left\{\chi(x-s)+\chi(x-n-s)+\dots\dots\dots\left(\overline{\overline{\tfrac{y-2}{t}}}+1\right)\text{terms}\right\}\\ +m_{z-2s}&\left\{\chi(x-2s)+\chi(x-n-2s)+\dots\left(\overline{\overline{\tfrac{y-3}{t}}}+1\right)\text{terms}\right\}\\ &+\text{\&c. } (t \text{ series}).\end{aligned}$$

Now we have

$$\overline{\overline{\frac{y}{t}}}=\frac{y}{t}t_y+\frac{y-1}{t}t_{y-1}+\dots\frac{y-t+1}{t}t_{y-t+1},$$

whence

$$\overline{\overline{\frac{y-1}{t}}}=\frac{y-1}{t}t_{y-1}+\frac{y-2}{t}t_{y-2}+\dots\frac{y-t}{t}t_y=\overline{\overline{\frac{y}{t}}}-t_y;$$

and similarly,

$$\overline{\overline{\frac{y-2}{t}}}=\frac{y-2}{t}t_{y-2}+\frac{y-3}{t}t_{y-3}+\dots\frac{y-t-1}{t}t_{y-1}=\overline{\overline{\frac{y}{t}}}-(t_y+t_{y-1}),$$

and so on. But by equation (25.), since $st=n$, we have

$$\overline{\overline{\frac{y}{t}}}=\frac{x}{n}-\frac{1}{n}\left\{0.n_x+1.n_{x-1}+2.n_{x-2}+\dots\overline{n-1}.n_{x-n+1}\right\}.$$

If therefore we put

$$\begin{aligned}\xi&=\frac{x}{n}\\ p_x&=\frac{1}{n}\left\{0.n_x+1.n_{x-1}+\dots\overline{n-1}.n_{x-n+1}\right\}-1\\ &=-\frac{1}{n}\left\{n.n_x+\overline{n-1}.n_{x-1}+\dots 1.n_{n-x+1}\right\}\\ q'_x&=n_x+n_{x-1}+\dots n_{x-s+1}\\ q''_x&=n_x+n_{x-1}+\dots n_{x-2s+1}\\ q'''_x&=n_x+n_{x-1}+\dots n_{x-3s+1}, \text{ \&c.,}\end{aligned}$$

we shall have

$$1+\overline{\overline{\frac{y-1}{t}}}=\xi-(p_x+q'_x);\quad 1+\overline{\overline{\frac{y-2}{t}}}=\xi-(p_x+q''_x);\ \text{\&c.,}$$

and therefore by equation (27.),

$$R=m_z.\frac{1-(1+\Delta)^{-\xi+p_x+q'_x}}{\Delta}\chi(x+n+n.0)$$

$$+m_{z-s}.\frac{1-(1+\Delta)^{-\xi+p_x+q''_x}}{\Delta}\chi(x+n-s+n.0)$$

$$+m_{z-2s}.\&c.+\&c.,$$

which resolves itself, by the transformation of equation (9.), into two sets of terms, $R'+R''$, viz.

$$R'=m_z.\frac{1-(1+\Delta)^{-\xi}}{\Delta}\chi(x+n+n.0)+m_{z-s}.\frac{1-(1+\Delta)^{-\xi}}{\Delta}\chi(x+n-s+n.0)+\&c. \quad . \quad . \quad (43.)$$

and

$$R''=m_z.\frac{1-(1+\Delta)^{p_x+q'_x}}{\Delta}\chi(n+n.0)+m_{z-s}.\frac{1-(1+\Delta)^{p_x+q''_x}}{\Delta}\chi(n-s+n.0)+\&c. \quad . \quad . \quad (44.)$$

(30.) If, in pursuance of the process followed in the development of Y, we put

$$X(x)=m_z.\chi(x+n)+m_{z-s}.\chi(x+n-s)+\&c. \ (t \text{ terms}),$$

(denoting also by $X_0(x)$, $X_1(x)$ &c., what this expression becomes when for z we put successively x, $x-1$, $x-2$, &c.), we shall have

$$R'=\frac{1-(1+\Delta)^{-\frac{x}{n}}}{\Delta}X(x+n.0)$$

$$=\frac{1}{n}\left\{\frac{x}{1}X(x)-\frac{x(x+n)}{1.2}\Delta 0.X'(x)+\left\{\frac{x(x+n)(x+2n)}{1.2.3}\Delta^2 0^2-\frac{x\overline{(x+n)}}{1.2}.n\Delta 0^2\right\}.\frac{X''(x)}{1.2}-\&c. \quad (45.)\right.$$

The whole assemblage of such terms, giving z all its values, from x to $x-m+1$, therefore will constitute a circulating function explicit in x, and which we shall denote by Z'.

As regards R'', since s and t are given numerically, it constitutes a periodic function with constant coefficients, to obtain which we have only to consider that, supposing any one of the exponents p_x+q_x to be represented by

$$a.n_x+b.n_{x-1}+c.n_{x-2}+\&c.,$$

we shall have by equation (16.),

$$\frac{1-(1+\Delta)^{p_x+q'_x}}{\Delta}=\frac{1-(1+\Delta)^a}{\Delta}.n_x+\frac{1-(1+\Delta)^b}{\Delta}.n_{x-1}+\&c.$$

(31.) In the particular case in which all the functions $\chi_0(x)$, $\chi_1(x)$, &c. are constant, we may consider them as being themselves the coefficients of a periodic function, such that

$$\chi_i=\chi_0.m_i+\chi_1.m_{i-1}+\ldots.\chi_{m-1}.m_{i-m+1},$$

so that if we should meet with such expressions as χ_m, χ_{m+1}, &c., they are to be taken as equivalent to χ_0, χ_1, &c., a mode of regarding a series of arbitrary constants occurring in a certain order which will tend greatly to simplify and add clearness to what follows. Now we have, generally χ_i being constant,

$$\frac{1-(1+\Delta)^h}{\Delta}\chi_i=-h.\chi_i.$$

Consequently, the terms R'_i and R''_i of R_i, corresponding to χ_i in the same way as R' and R'' in general to χ, will become

$$R'_i=\{m_z+m_{z-s}+\ldots\ldots m_{z-ts+s}\}\cdot\frac{x}{n}\cdot\chi_i$$

$$\begin{aligned}R''_i=&-\{m_z+m_{z-s}+\ldots\ldots m_{z-ts+s}\}\cdot p_x\cdot\chi_i\\&-m_z\{n_x+n_{x-1}+\ldots\ldots n_{x-s+1}\}\cdot\chi_i\\&-m_{z-s}\{n_x+n_{x-1}+\ldots\ldots n_{x-2s+1}\}\cdot\chi_i-\text{\&c.},\end{aligned}$$

in which it will be recollected that $z=x-i$.

(32.) Now if v be the greatest common measure of m and s (v being 1, if these numbers be prime to each other), we have

$$m_z+m_{z-s}+\ldots\ldots m_{z-ts+s}=v_z,$$

and consequently the value of R_i or R_i+R_i becomes

$$\begin{aligned}R_i=\left(\frac{x}{n}-p_x\right)\cdot\chi_i\cdot v_{x-i}&-m_{x-i}\{n_x+\ldots\ldots n_{x-s+1}\}\cdot\chi_i\\&-m_{x-s-i}\{n_x+\ldots\ldots n_{x-2s+1}\}\cdot\chi_i\\&-\text{\&c.}\end{aligned}$$

(33.) Assembling together similar results for R_0, R_1, ... R_{m-1}, we have

$$\begin{aligned}Z=&\{\chi_0\cdot v_x+\chi_1\cdot v_{x-1}+\ldots\ldots\chi_{m-1}\cdot v_{x-m+1}\}\cdot\left(\frac{x}{n}-p_x\right)\\&-\{n_x+\ldots\ldots n_{x-s+1}\}\{\chi_0\cdot m_x+\chi_1\cdot m_{x-1}+\ldots\ldots\chi_{m-1}\, m_{x-m+1}\}\\&-\{n_x+\ldots\ldots n_{x-2s+1}\}\{\chi_0\cdot m_{x-s}+\chi_1\cdot m_{x-s-1}+\ldots\ldots\chi_{m-1}\cdot m_{x-s+1}\}\\&-\{n_x+\ldots\ldots n_{x-3s+1}\}\{\chi_0\cdot m_{x-2s}+\chi_1\cdot m_{x-2s-1}+\ldots\ldots\chi_{m-1}\cdot m_{x-2s+1}\}\\&-\text{\&c.}\end{aligned}$$

Now because m and s have v for a common measure, and that $n=ts$ is the first multiple of s, which is also a multiple of m, it follows that $n=\frac{s}{v}.m$, $\frac{s}{v}$ being an integer. Hence we have by equation (22.),

$$\begin{aligned}m_x&=n_x+n_{x-m}+n_{x-2m}\ldots\ldots+n_{x-n+m}\\m_{x-1}&=n_{x-1}+n_{x-m-1}+\ldots\ldots n_{x-n+m-1}\\\text{\&c.}&=\text{\&c.}\end{aligned}$$

Substituting these therefore, and so arranging the terms that n_x shall always stand first, the series within the brackets on the right-hand, in the expression for Z, become respectively

$$\begin{aligned}&\chi_0\cdot n_x+\chi_1\cdot n_{x-1}+\ldots\ldots\chi_{n-1}\cdot n_{x-n+1}\\&\chi_{m-s}\cdot n_x+\chi_{m-s+1}\cdot n_{x-1}+\ldots\ldots\chi_{m-s+n-1}\cdot n_{x-n+1}\\&\chi_{m-2s}\cdot n_x+\chi_{m-2s+1}\cdot n_{x-1}+\ldots\ldots\chi_{m-2s+n-1}\cdot n_{x-n+1},\ \text{\&c.},\end{aligned}$$

which being multiplied by their respective coefficients, $n_x+n_{x-1}+$ &c., we get for Z as follows:—

$$Z=\frac{x}{n}\left\{\chi_0.v_x+\chi_1.v_{x-1}+\ldots.\chi_{m-1}.v_{x-m+1}\right\} \quad . \quad . \quad . \quad . \quad . \quad (46.)$$
$$-p_x.\{\chi_0.v_x+\chi_1.v_{x-1}+\ldots\chi_{m-1}.v_{x-m+1}\}$$
$$-\{\chi_0.n_x+\chi_1.n_{x-1}+\chi_2.n_{x-2}+\ldots.(s \text{ terms})\}$$
$$-\{\chi_{m-s}.n_x+\chi_{m-s+1}.n_{x-1}+\ldots\ldots\ldots(2s \text{ terms})\}$$
$$-\{\chi_{m-2s}.n_x+\chi_{m-2s+1}.n_{x-1}+\ldots\ldots(3s \text{ terms})\}$$
$$-\text{ \&c.}$$

The first line of this is a circulating function, linear in x in all cases except when $v=1$, or m and s are prime to each other, in which case it loses its circulating character and becomes simply

$$\frac{x}{n}(\chi_0+\chi_1+\chi_2+\ldots\chi_{m-1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47.)$$

As regards the second line, we have

$$-p_x=\frac{1}{n}\left\{n.n_x+\overline{n-1}.n_{x-1}+\ldots 1.n_{x-n+1}\right\};$$

and since n is a multiple of s, and therefore of v, the multiplier within the brackets is readily reduced to a periodic function having n for its period, such as $a.n_x+b.n_{x-1}+$ &c., which, multiplied by $-p_x$, gives

$$\frac{1}{n}\left\{na.n_x+\overline{n-1}.b.n_{x-1}+\text{\&c.}\right\},$$

except when $v=1$, in which case this expression reduces itself to

$$\frac{1}{n}(\chi_0+\chi_1+\ldots.\chi_{m-1})\{n.n_x+\overline{n-1}.n_{x-1}+\ldots.1.n_{x-n+1}\} \quad . \quad . \quad . \quad . \quad (48.)$$

(34.) To apply the foregoing formulæ to the expression of particular cases as $\overset{2}{\Pi}(x)$, $\overset{3}{\Pi}(x)$, $\overset{4}{\Pi}(x)$, &c., we begin with $\overset{1}{\Pi}(x)=1$. Therefore, to find $\overset{2}{\Pi}(x)$, we have

$$\varphi(x)=1, \quad \varphi'(x) \text{ \&c.}=0, \quad \psi_1(s)=s-1, \quad s=2;$$

consequently

$$X=\frac{x}{2}; \quad \nabla(s+0.s)=-\frac{1}{2}; \quad Y=-\frac{1}{2}.2_{x-1},$$

and therefore

$$\overset{2}{\Pi}(x)=\frac{1}{2}(x-2_{x-1}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49.)$$

(35.) For the case of $s=3$, we have

$$\varphi(x)=\frac{x}{2}; \quad \varphi'(x)=\frac{1}{2};$$

$$\psi_1(s)=\frac{s-1}{2}=1; \quad \psi_1'(s)=\frac{1}{2}; \quad \psi_2(s)=\frac{\overline{s-1}+s}{2}=\frac{5}{2}; \quad \psi_2'(s)=1;$$

and therefore

$$X=\frac{1}{6}\left\{x(x+2)-\frac{x(x+3)}{2}\right\}=\frac{x^2+x}{12}.$$

For Y we have

$$\Psi(s)=0.3_x+1.3_{x-1}+\frac{5}{2}.3_{x-2}$$

$$\Psi'(s)=0.3_x+\frac{1}{2}.3_{x-1}+3_{x-2}$$

whence by equation (37.),

$$Y=-\frac{1}{12}\left\{2.6_{x-1}+6.6_{x-2}+2.6_{x-4}+6.6_{x-5}\right\}.$$

For Z we have

$$Q_x=-\frac{1}{2}.2_{x-1}\quad Q_{x-1}=-\frac{1}{2}2_x$$

whence

$$m=2,\ s=3,\ ms=6,\ v=1,\ \chi_0=-\frac{1}{2},\ \chi_1=0$$

and by (45.), (46), (47.),

$$Z=-\frac{x}{12}-\frac{1}{12}\left\{6.6_x+5.6_{x-1}+4.6_{x-2}+3.6_{x-3}+2.6_{x-4}+1.6_{x-5}\right\}$$

$$+\frac{6}{12}\left\{6_x+6_{x-2}\right\}+\frac{6}{12}\left\{6_{x-1}+6_{x-3}+6_{x-5}\right\}$$

$$=-\frac{x}{12}+\frac{1}{12}\left\{6_{x-1}+2.6_{x-2}+3.6_{x-3}-2.6_{x-4}+5.6_{x-5}\right\}$$

adding all which parts together, we have X+Y+Z, or

$$\overset{3}{\Pi}(x)=\frac{1}{12}\left\{x^2-6_{x-1}-4.6_{x-2}+3.6_{x-3}-4.6_{x-4}-6_{x-5}\right\}. \qquad (50.)$$

(36.) When $s=4$, we have, therefore,

$$\phi(x)=\frac{x^2}{12},\quad \phi'(x)=\frac{2x}{12},\quad \phi''(x)=\frac{2}{12}$$

and, therefore, by equation (35.),

$$X=\frac{1}{48}\left\{x(x+3)^2-\frac{x(x+4)}{2}.2(x+3)+\left\{\frac{x(x+4)(x+8)}{6}.2-\frac{x(x+4)}{2}.4\right\}.\frac{2}{1.2}\right\}=\frac{x^3+3x^2-x}{144}.$$

For Y we have

$$\psi_1(s)=\frac{(s-1)^2}{12}=\frac{9}{12};\quad \psi_1'(s)=\frac{2(s-1)}{12}=\frac{6}{12};\quad \psi_1''(s)=\frac{2}{12}$$

$$\psi_2(s)=\frac{(s-1)^2+s^2}{12}=\frac{25}{12};\quad \psi_2'(s)=\frac{2(s-1)+2s}{12}=\frac{14}{12};\quad \psi_2''(s)=\frac{4}{12}$$

$$\psi_3(s)=\frac{(s-1)^2+s^2+(s+1)^2}{12}=\frac{50}{12};\quad \psi_3'(s)=\frac{2(s-1)+2s+2(s+1)}{12}=\frac{24}{12};\quad \psi_3''(s)=\frac{6}{12}$$

$$\Psi(s)=\frac{1}{12}\left\{0.4_x+9.4_{x-1}+25.4_{x-2}+50.4_{x-3}\right\}$$

$$\Psi'(s)=\frac{1}{12}\left\{0.4_x+6.4_{x-1}+14.4_{x-2}+24.4_{x-3}\right\}$$

$$\Psi''(s)=\frac{1}{12}\left\{0.4_x+2.4_{x-1}+4.4_{x-2}+6.4_{x-3}\right\},$$

whence by equation (37.),

$$Y=-\frac{1}{48}\left\{\Psi(s)-\frac{3}{2}\Psi'(s)+\frac{1}{2}\Psi''(s)\right\}$$

$$=-\frac{1}{48}\left\{0.4_x+1.4_{x-1}+6.4_{x-2}+17.4_{x-3}\right\}$$

$$=-\frac{1}{144}\left\{0.12_x+3.12_{x-1}+18.12_{x-2}+51.12_{x-3}+0.12_{x-4}+3.12_{x-5}+18.12_{x-6}+\right.$$
$$\left.+51.12_{x-7}+0.12_{x-8}+3.12_{x-9}+18.12_{x-10}+51.12_{x-11}\right\}.$$

Lastly, for Z we have

$$Q_x\ =\left\{-0.6_x-1.6_{x-1}-4.6_{x-2}+3.6_{x-3}-4.6_{x-4}-1.6_{x-5}\right\}.\frac{1}{12}$$

$$Q_{x-1}=\left\{-1.6_x-0.6_{x-1}-1.6_{x-2}-4.6_{x-3}+3.6_{x-4}-4.6_{x-5}\right\}.\frac{1}{12},$$

consequently

$$m=6,\ s=4,\ v=2,\ t=3,\ n=st=12$$

$$\chi_0=-\frac{1}{12},\ \chi_1=0,\ \chi_2=-\frac{1}{12},\ \chi_3=-\frac{4}{12},\ \chi_4=+\frac{3}{12},\ \chi_5=-\frac{4}{12},$$

and therefore, by equation (46.), which gives the value of Z in this case, putting Z' for the first line of Z, Z'' for the second and Z''' for the rest,

$$Z'=\frac{x}{144}\left\{-1.2_x-0.2_{x-1}-1.2_{x-2}-4.2_{x-3}+3.2_{x-4}-4.2_{x-5}\right\}$$

$$=\frac{x}{144}\left\{(-1-1+3).2_x+(0-4-4).2_{x-1}\right\}$$

$$=\frac{x}{144}\left\{2_x-8.2_{x-1}\right\}=\frac{x}{144}-\frac{9x}{144}.2_{x-1}$$

$$Z''=(2_x-8.2_{x-1}).\frac{1}{144}\left\{12.12_x+11.12_{x-1}+10.12_{x-2}+\ldots..1.12_{x-11}\right\},$$

which, putting for 2_x and 2_{x-1} their values

$$12_x+12_{x-2}+12_{x-4}\ldots..+12_{x-10},$$

and

$$12_{x-1}+12_{x-3}+\ldots.12_{x-11}$$

becomes

$$Z''=\frac{1}{144}\left\{12.12_x-88.12_{x-1}+10.12_{x-2}-72.12_{x-3}+8.12_{x-4}-56.12_{x-5}+6.12_{x-6}-\right.$$
$$\left.-40.12_{x-7}+4.12_{x-8}-24.12_{x-9}+2.12_{x-10}-8.12_{x-11}\right\};$$

and lastly,

$$Z'''=\frac{1}{12}\left\{1.12_x+0.12_{x-1}+1.12_{x-2}+4.12_{x-3}\right\}$$
$$+\frac{1}{12}\left\{1.12_x+4.12_{x-1}-3.12_{x-2}+4.12_{x-3}+1.12_{x-4}+0.12_{x-5}+1.12_{x-6}+4.12_{x-7}\right\}$$
$$+\frac{1}{12}\left\{-3.12_x+4.12_{x-1}+1.12_{x-2}+0.12_{x-3}+1.12_{x-4}+4.12_{x-5}-3.12_{x-6}+4.12_{x-7}\right.$$
$$\left.+1.12_{x-8}+0.12_{x-9}+1.12_{x-10}+4.12_{x-11}\right\}$$

$$=\frac{12}{144}\Big\{-12_x+8.12_{x-1}-1.12_{x-2}+8.12_{x-3}+2.12_{x-4}+4.12_{x-5}-2.12_{x-6}+8.12_{x-7}+$$
$$+1.12_{x-8}+0.12_{x-9}+1.12_{x-10}+4.12_{x-11}\Big\}.$$

And assembling these several portions, $X+Y+(Z'+Z''+Z''')$, we get

$$\overset{4}{\Pi}(x)=\frac{1}{144}\Big\{x^3+3x^2-9x.2_{x-1}\Big\}+\frac{1}{144}\Big\{0.12_x+5.12_{x-1}-20.12_{x-2}-27.12_{x-3}+32.12_{x-4}$$
$$-11.12_{x-5}-36.12_{x-6}+5.12_{x-7}+16.12_{x-8}-27.12_{x-9}-4.12_{x-10}-11.12_{x-11}\Big\}. \quad (51.)$$

(37.) Proceeding now to the case where $s=5$, we have

$$\varphi(x)=\frac{x^3+3x^2}{144},\ \varphi'(x)=\frac{3x^2+6x}{144},\ \varphi''(x)=\frac{6x+6}{144},\ \varphi'''(x)=\frac{6}{144},$$

whence

$$\varphi(x+4)=\frac{(x+4)^2(x+7)}{144},\ \varphi'(x+4)=\frac{3(x+4)(x+6)}{144},\ \varphi''(x+4)=\frac{6(x+5)}{144},\ \varphi'''(x+4)=\frac{6}{144},$$

and executing the reductions, arising from the substitution of these in equation (35.), we find

$$X=\frac{x^4+10x^3+19x^2-22x}{2880}.$$

Again for Y we have

$$\psi_1(5)=\varphi(4);\ \psi_2(5)=\varphi(4)+\varphi(5);\ \psi_3(5)=\varphi(4)+\varphi(5)+\varphi(6),\ \&c.,$$

whence

$$\psi_1(5)=\frac{112}{144};\ \psi_2(5)=\frac{312}{144};\ \psi_3(5)=\frac{636}{144};\ \psi_4(5)=\frac{1126}{144}$$

$$\psi_1'(5)=\frac{72}{144};\ \psi_2'(5)=\frac{177}{144};\ \psi_3'(5)=\frac{321}{144};\ \psi_4'(5)=\frac{510}{144}$$

$$\psi_1''(5)=\frac{30}{144};\ \psi_2''(5)=\frac{66}{144};\ \psi_3''(5)=\frac{108}{144};\ \psi_4''(5)=\frac{156}{144}$$

$$\psi_1'''(5)=\frac{6}{144};\ \psi_2'''(5)=\frac{12}{144};\ \psi_3'''(5)=\frac{18}{144};\ \psi_4'''(5)=\frac{24}{144}$$

$$\Psi(5)=\frac{1}{144}\Big\{0.5_x+112.5_{x-1}+312.5_{x-2}+636.5_{x-3}+1126.5_{x-4}\Big\}$$

$$\Psi'(5)=\frac{1}{144}\Big\{0.5_x+72.5_{x-1}+177.5_{x-2}+321.5_{x-3}+510.5_{x-4}\Big\}$$

$$\Psi''(5)=\frac{1}{144}\Big\{0.5_x+30.5_{x-1}+66.5_{x-2}+108.5_{x-3}+156.5_{x-4}\Big\}$$

$$\Psi'''(5)=\frac{1}{144}\Big\{0.5_x+6.5_{x-1}+12.5_{x-2}+18.5_{x-3}+24.5_{x-4}\Big\}$$

$$Y=-\frac{4}{2880}\Big\{\Psi(5)-2.\Psi'(5)+\Psi''(5)+\tfrac{2}{3}\Psi'''(5)\Big\}$$

$$=-\frac{1}{2880}\Big\{0.5_x+8.5_{x-1}+128.5_{x-2}+456.5_{x-3}+1112\cdot5_{x-4}\Big\}.$$

(38.) For Z we have

$$Q_x \;=-\frac{9x}{144}.2_{x-1}+\frac{1}{144}\left\{0.12_x+5.12_{x-1}+\ldots..-11.12_{x-11}\right\}$$

$$Q_{x-1}=-\frac{9(x-1)}{144}.2_x+\frac{1}{144}\left\{-11.12_x+0.12_{x-1}+5.12_{x-2}\ldots.-4.12_{x-11}\right\}.$$

It will be convenient to separate this into two parts, viz.

$$Q'_{x-1}=-\frac{9(x-1)}{144}.2_x,$$

and

$$Q''_{x-1}=\frac{1}{144}\left\{-11.12_x+0.12_{x-1}+\&c.\right\}.$$

First, then, for Q'_{x-1}, proceeding as in article 30, we have

$$z=x,\ \chi_0(x)=-\frac{9}{144}(x-1)\,;\ \chi_1(x)=0.$$

$$X(x)=2_x.\chi_0(x+10)+2_{x-1}\chi_0(x+5)$$

$$=-\frac{9}{144}\left\{(x+9).2_x+(x+4).2_{x-1}\right\}$$

$$X'(x)=-\frac{9}{144}(2_x+2_{x-1})=-\frac{9}{144},$$

whence Z' consisting now of the single term R',

$$Z'=-\frac{9}{1440}\left\{x^2+x(9.2_x+4.2_{x-1})-\frac{x(x+10)}{2}\right\}$$

$$=-\frac{9}{2880}\left\{x^2-2x+10x.2_x\right\}.$$

As regards the other portion of Z, which in this case is R'', it has for its expression

$$-\frac{9}{144}2_x.\frac{1-(1+\Delta)^{p_x+q'_x}}{\Delta}(9+10.0)+2_{x-1}.\frac{1-(1+\Delta)^{p_x+q''_x}}{\Delta}(4+10.0).$$

Now, whatever be h and c, we have always

$$\frac{1-(1+\Delta)^h}{\Delta}(c+10.0)=(5-c)h-5h^2.$$

In this, if we write for h successively $h'=p_x+q'_x$ and $h''=p_x+q''_x$, and for c, 9 and 4, we find

$$Z''=+\frac{9}{144}\left\{2_x(4h'+5h'^2)-2_{x-1}(h''-5h''^2)\right\}.$$

But since $x=10$ and $s=5$, we have

$$h'=\frac{1}{10}\left\{0.10_x+1.10_{x-1}+2.10_{x-2}+3.10_{x-3}+4.10_{x-4}-5.10_{x-5}-4.10_{x-6}-3.10_{x-7}-2.10_{x-8}-1.10_{x-9}\right\}$$

$$h'=\frac{1}{10}\left\{0.10_x+1.10_{x-1}+2.10_{x-2}+\ \ldots\ldots\ldots\ldots\ldots\ldots\ +9.10_{x-9}\right\}.$$

Substituting which in Z'' and employing the property of equation (14.) for the computation of the coefficients, we find

$$Z''=\frac{9}{2880}\left\{\begin{array}{l}2_x\{0.10_x+9.10_{x-1}+20.10_{x-2}+33.10_{x-3}+48.10_{x-4}-15.10_{x-5}-16.10_{x-6}-15.10_{x-7}-12.10_{x-8}-7.10_{x-9}\}\\ +2_{x-1}\{0.10_x-1.10_{x-1}+0.10_{x-2}+3.10_{x-3}+8.10_{x-4}+15.10_{x-5}+24.10_{x-6}+35.10_{x-7}+48.10_{x-8}+63.10_{x-9}\}\end{array}\right\}$$

$$=\frac{9}{2880}\left\{0.10_x-1.10_{x-1}+20.10_{x-2}+3.10_{x-3}+48.10_{x-4}+15.10_{x-5}-16.10_{x-6}+35.10_{x-7}-12.10_{x-8}+63.10_{x-9}\right\}.$$

(39.) Finally, we have to consider the portion Z''' of Z originating in Q''_{x-1}, in which the values of χ_0, χ_1, &c. are given by the equation

$$\chi_i=\frac{1}{144}\left\{-11.12_i+0.12_{i-1}+5.12_{i-2}-20.12_{i-3}\ldots\ldots-4.12_{i-11}\right\},$$

the coefficients being those of equation (51.) in their order of circulation. We have also, since in this case $m=12$, $s=5$, and therefore prime to each other, $v=1$, $t=12$, $n=60$. Whence

$$\chi_0.v_x+\chi_1v_{x-1}+\ldots\ldots\chi_{m-1}v_{x-m+1}=\chi_0+\chi_1+\ldots\chi_{m-1}=-\frac{78}{144},$$

and therefore

$$Z'''=-\frac{26x}{2880}-\frac{26}{2880}\left\{60.60_x+59.60_{x-1}+58.60_{x-2}+\ldots\ldots1.60_{x-59}\right\}$$

$$-\frac{20}{2880}\left\{-11.60_x+0.60_{x-1}+5.60_{x-2}-20.60_{x-3}-27.60_{x-4}\right\}$$

$$-\frac{20}{2880}\left\{-36.60_x+5.60_{x-1}+16.6_{x-2}-27.60_{x-3}-4.60_{x-4}-11.60_{x-5}+0.60_{x-6}+5.60_{x-7}-20.60_{x-8}-27.60_{x-9}\right\}$$

$$-\frac{20}{2880}\left\{+5.60_x-20.60_{x-1}-27.60_{x-2}\ldots\ldots-27.60_{x-14}\right\}$$

$$-\frac{20}{2880}\left\{+16.60_x-27.60_{x-1}-4.60_{x-2}\ldots\ldots-27.60_{x-19}\right\}$$

$$-\frac{20}{2880}\left\{-27.60_x+32.60_{x-1}-11.60_{x-2}\ldots\ldots-27.60_{x-24}\right\}$$

$$-\frac{20}{2880}\left\{-4.60_x-11.60_{x-1}+0.60_{x-2}\ldots\ldots-27.60_{x-29}\right\}$$

$$-\frac{20}{2880}\left\{-11.60_x-36.60_{x-1}+5.60_{x-2}\ldots\ldots-27.60_{x-34}\right\}$$

$$-\frac{20}{2880}\left\{0.60_x+5.60_{x-1}-20.60_{x-2}\ldots\ldots-27.60_{x-39}\right\}$$

$$-\frac{20}{2880}\left\{+5.60_x+16.60_{x-1}-27.60_{x-2}\ldots\ldots-27.60_{x-44}\right\}$$

$$-\frac{20}{2880}\left\{-20.60_x-27.60_{x-1}+32.60_{x-2}\ldots\ldots-27.60_{x-49}\right\}$$

$$-\frac{20}{2880}\left\{-27.60_x-4.60_{x-1}-11.60_{x-2}\ldots\ldots-27.60_{x-54}\right\}$$

$$-\frac{20}{2880}\left\{+32.60_x-11.60_{x-1}-36.60_{x-2}\ldots\ldots-27.60_{x-59}\right\}$$

Assembling, finally, the several portions, X, Y, Z', Z'', Z''', of which $\overset{5}{\Pi}(x)$ consists, and reducing those periodic functions, which have 5 and 10 for their period, to a period of 60, we see

$$\overset{5}{\Pi}(x)=\frac{1}{2880}\left\{x^4+10x^3+10x^2-30x-90x.2_x\right\}$$
$$+\frac{1}{2880}\left\{0.60_x+9.60_{x-1}+104.60_{x-2}-387.60_{x-3}-576.60_{x-4}+905.60_{x-5}\right.$$
$$-216.60_{x-6}-351.60_{x-7}-256.60_{x-8}+9.60_{x-9}+360.60_{x-10}-31.60_{x-11}$$
$$-576.60_{x-12}+9.60_{x-13}+104.60_{x-14}+225.60_{x-15}-576.60_{x-16}+329.60_{x-17}$$
$$-216.60_{x-18}-351.60_{x-19}+320.60_{x-20}+9.60_{x-21}-216.60_{x-22}-31.60_{x-23}$$
$$-576.60_{x-24}+585.60_{x-25}+104.60_{x-26}-351.60_{x-27}-576.60_{x-28}+329.60_{x-29}$$
$$+360.60_{x-30}-351.60_{x-31}-256.60_{x-32}+9.60_{x-33}-216.60_{x-34}+545.60_{x-35}$$
$$-576.60_{x-36}+9.60_{x-37}+104.60_{x-38}-351.60_{x-39}+0.60_{x-40}+329.60_{x-41}$$
$$-216.60_{x-42}-351.60_{x-43}-256.60_{x-44}+585.60_{x-45}-216.60_{x-46}-31.60_{x-47}$$
$$-576.60_{x-48}+9.60_{x-49}+680.60_{x-50}-351.60_{x-51}-576.60_{x-52}+329.60_{x-53}$$
$$\left.-216.60_{x-54}+225.60_{x-55}-256.60_{x-56}+9.60_{x-57}-216.60_{x-58}-31.60_{x-59}\right\}. \quad (52.)$$

(40.) The periodic function $0.60_x+\ldots.$&c. may be somewhat simplified by resolving it into the sum of three others, having respectively 10, 20 and 30 for their periods. For on inspecting its coefficients, we find that the differences of any two, distant from each other by 30, are alternately +360 and −360. Now if we suppose, generally, any such function as

$$a_0.60_x+a_1.60_{x-1}+\&c.$$

to be made up of the sum of three others,

$$p_0.30_x+p_1.30_{x-1}+\&c.$$
$$q_0.20_x+q_1.20_{x-1}+\&c.$$
$$r_0.10_x+r_1.10_{x-1}+\&c.,$$

we shall have, supposing i any number of the series 0, 1, 2, ... 9,

$$p_i+q_i+r_i=a_i, \quad p_{i+10}+q_{i+10}+r_i=a_{i+10}, \quad p_{i+20}+q_i+r_i=a_{i+20}$$
$$p_i+q_{i+10}+r_i=a_{i+30}, \quad p_{i+10}+q_i+r_i=a_{i+40}, \quad p_{i+20}+q_{i+10}+r_i=a_{i+50}$$

which give the following equations of condition among the coefficients a,

$$a_{30+i}-a_i=-(a_{40+i}-a_{10+i})=a_{50+i}-a_{20+i}$$

And if these be satisfied (as in this case they are), we have only further to establish the following relations between p, q, r, viz.

$$p_i+q_i+r_i=a_i$$
$$p_{i+10}=p_i+(a_{i+10}-a_{i+30})\,; \quad p_{i+20}=p_i+(a_{i+20}-a_i)$$
$$q_{i+10}=q_i+(a_{i+30}-a_i).$$

Among the sixty coefficients therefore which this assumption places at our disposal, twenty remain arbitrary, and may be put =0.

Suppose, for example,

$$q_i=0, \quad p_{i+20}=0,$$

which give

$$p_i = a_i - a_{i+20}, \quad p_{i+10} = (a_i - a_{i+30}) + (a_{i+10} - a_{i+20})$$

$$q_{i+10} = a_{i+30} - a_i, \quad r_i = a_{i+20}.$$

These being calculated, the function $0.60_x + 9.60_{x-1} + \&c.$ reduces itself to the following, which seems the simplest form it admits:

$$-320\{30_x - 30_{x-2} + 30_{x-3} - 30_{x-5} + 30_{x-6} - 30_{x-8} + 30_{x-9} + 30_{x-10} - 30_{x-11} + 30_{x-13}$$
$$-30_{x-14} + 30_{x-16} - 30_{x-17} + 30_{x-19}\}$$
$$+360\{20_{x-10} + 20_{x-11} + 20_{x-12} + 20_{x-13} + 20_{x-14} + \ldots\ldots 20_{x-19}\}$$
$$+\{320.10_x + 9.10_{x-1} - 216.10_{x-2} - 31.10_{x-3} - 576.10_{x-4} + 585.10_{x-5} + 104.10_{x-6}$$
$$-351.10_{x-7} - 576.10_{x-8} + 329.10_{x-9}. \quad \ldots \ldots \quad (53.)$$

(41.) The problem, "In how many ways can a given number be constructed," is reduced by the author of a short but interesting paper in the Cambridge Mathematical Journal, iv. p. 87*, to the integration of the equations of differences

$$u_{x,y} = u_{x-y,\,x+y} \text{ and } u_{x,y} - u_{x-y,\,y} = u_{x-1,\,y-1},$$

which last equation corresponds to the case where it is required to find in how many ways x can be composed of numbers none greater and not all less than y. The analogy of this problem with that here treated is obvious, the function $u_{x,y}$ being in effect identical with that which in the above notation would be expressed by $\overset{y}{\Pi}(x)$, y corresponding to our s. Accordingly, as far as $y=4$, to which limit only the inquiry is there extended, the results are identical (the mode of expression excepted) with those of our equations (49.), (50.), (51.)†. The method there pursued (by the successive integration of equations of differences) would of course continue to afford similar results, but without some systematic processes of notation, transformation and reduction, such as those delivered in the foregoing pages, would speedily become too complicated to be followed out, though the sort of form which would ultimately be assumed by the result seems to have been clearly apprehended. Observing that in the cases of $y=2, 3, 4$, the results express in fact *the nearest integers* to certain rational fractions, such as $\frac{x^2}{12}$ in the case of $y=3$, $\frac{x^3+3x^2}{2}$ (x even) and $\frac{x^3+3x^2-9x}{2}$ (x odd) when $y=4$, it is suggested that "probably this simple species of description might be continued." This, on examination of the value above given, when y or $s=5$, appears to be the case, but for higher values it will be necessary to enlarge the terms of the description, so as to take in circulating functions of higher orders, and with more complicated coefficients. To make this apparent, suppose $s=6$. Then, without going into the whole calculation (which however would not be materially more complicated than for $s=5$, and would lead, as in that case, to a final period of 60, only not reducible to the sum

* It bears no name, but I have reason to believe it to be the production of Professor DeMorgan.

† Mr. Warburton has also obtained expressions for the number of partitions as far as 4, and his results, *mutatis mutandis*, agree with the above.

of lower periods), it is easy to see, that besides a non-periodic portion of 5 dimensions in x, and a periodic one with 60 constant coefficients, there will also be a circulating portion of the form

$$a_x.6_x+b_x.6_{x-1}+\dots f_x.6_{x-5},$$

whose coefficients may rise to the second dimension in x. In fact, if we execute the calculation of this portion by the foregoing processes, we find for the values of the coefficients

$$a_x=e_x=0\,;\quad b_x=d_x=-\frac{4500x^2+15750x}{172800}\,;\quad c_x=\frac{3209x}{172800}.$$

With regard to the constant coefficients of the periodic portion, it is easy to see, from the manner of their formation, that they must all fall very far short, in numerical magnitude, of the half of 172800, so that the whole effect of this periodical part does, in effect, go to adjust the final value to *the nearest integer* of the rational fraction arising from the assemblage of all the terms in x, and a similar reasoning will apply in all cases.

(42.) The number of partitions of which a given number x is susceptible, admitting 0 into them as a component part, is the sum of the number of 1-partitions, bipartitions, tripartitions .... up to s-partitions. It may therefore be found, by adding together all the values of $\overset{s}{\Pi}(x)$, from $s=1$ to $s=s$ inclusive. But it may also be obtained by formulæ in all respects similar to those above demonstrated; for if we take $\Pi_s(x)$ to represent this species of partition, we have, if $s=1$, $\Pi_1(x)=1$ as before. For $s=2$ the partitions stand $0, x$; $1, x-1$; $2, x-2$; ... $\left(\frac{x}{2}+1\right)$ terms, that is,

$$\Pi_2(x)=\Pi_1(x)+\Pi_1(x-2)+\dots\left(\frac{x}{2}+1\right)\text{terms.}$$

Similarly,

$$\Pi_3(x)=\Pi_2(x)+\Pi_2(x-3)+\dots.\left(\frac{x}{3}+1\right)\text{terms,}$$

and so on to

$$\Pi_s(x)=\Pi_{s-1}(x)+\Pi_{s-1}(x-s)+\dots.\left(\frac{x}{s}+1\right)\text{terms,}$$

of which the formulæ of (30.) and (31.), duly applied, give the value

$$\Pi_s(x)=\frac{1-(1+\Delta)^{-\frac{x}{s}}}{\Delta}\Pi_{s-1}(x+s+s.0)+\frac{1-(1+\Delta)_q}{\Delta}\Pi_{s-1}(s+s.0),$$

where

$$q=-\frac{1}{s}\left\{\overline{s-1}\,.s_{x-1}+\overline{s-2}\,.s_{x-2}+\dots.1.s_{x-s+1}\right\},$$

which, developed, affords a calculable value of the function in question.

Collingwood, April 17, 1850.